Observable and Reachable Dual Systems for Discrete-Time Surrogate Singular Linear Switched Systems

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***Abstract*—The duality between observability and reachability allows results and analysis methods for one property to be directly applied to the other, simplifying the design and understanding of control systems. This paper extends the investigation of this duality property from ordinary linear systems to surrogate singular linear switched systems in which each operational mode may be nonsingular or singular. The study focuses on the discrete-time domain. The dual system for such a class of systems is established. The results showed that the dual system cannot be derived via the "naive" dual used in ordinary systems. Instead, reversing the switching signal and switching the consistency space are needed, highlighting the implications for system analysis and design in the presence of mode-dependent singularities.**

***Index Terms*—Surrogate singular linear switched systems; Observability–reachability duality; Dual system construction.**

I. INTRODUCTION

Switched systems constitute an important class of hybrid dynamical systems in which the evolution of the state is governed by a family of subsystems and a switching signal selecting the active mode at each time instant [1]. Such systems naturally arise in a wide range of applications, including power electronics [2], [3], [4], networked control systems [5], [6], [7], multi-agent coordination [8], [9], [10] and cyber–physical systems [11], [12], [13], among others. In many of applications, the system dynamics are not only mode-dependent but may also exhibit algebraic constraints, leading to singular (descriptor) switched systems [14], [15], [16]. The presence of switching and singularity significantly enriches the system behavior, while at the same time posing substantial theoretical and practical challenges [17], [18].

Among the fundamental system-theoretic properties, reachability and observability play a central role in analysis, controller design, and state estimation. For classical linear time-invariant systems, the duality between reachability and observability is well established [19], [20]. This duality provides deep insights into system structure, enables the transfer of results between control and estimation problems, and simplifies both analysis and synthesis. In discrete-time systems, these properties admit elegant algebraic characterizations and strong geometric interpretations.

However, when extending these concepts to linear switched systems, the situation becomes considerably more intricate. The interaction between switching signals and system dynamics introduces nontrivial dependencies that invalidate many direct extensions of classical results. In particular, reachability and observability may depend not only on system matrices but also on admissible switching sequences, dwell-time constraints, or mode activation patterns [21]. While significant progress has been made in understanding these properties for nonsingular switched systems, a comprehensive and systematic treatment of their duality remains incomplete.

The problem becomes even more challenging in the context of singular linear switched systems. Descriptor dynamics introduce impulsive behavior, algebraic constraints, and consistency conditions on initial states, which fundamentally alter the nature of reachability and observability. In discrete time, singular systems require special care, as the distinction between dynamic and algebraic components directly affects reachability and state reconstruction (observability). Although reachability and observability of singular systems have been studied in the non-switched setting, their extension to switched descriptor systems is far from straightforward.

Motivated by these challenges, this paper focuses on finding the dual system for discrete-time surrogate singular linear switched systems. By examining the algebraic and geometric properties of the reachability and observability properties of those systems, precise duality results and conditions under which classical duality principles can be preserved or must be modified are established. This opens new knowledge in the structural relationship between reachability and observability in the presence of switching and singularity.

Understanding whether, and under what conditions, such a duality holds is not only of theoretical interest but also of practical importance. Furthermore, a rigorous duality framework allows one to derive observability results from reachability analysis (and vice versa), reduces redundancy in system-theoretic investigations, and supports the development of unified tools for control and estimation in hybrid systems. In doing so, this work contributes to a deeper understanding of fundamental system properties and lays groundwork for future research on control, estimation, and optimization of complex hybrid systems.

II. PROBLEM SETUP

We consider in this paper a discrete-time Inhomogeneous Singular Linear Switched System (InhSLSS) of the form

$$E_{\sigma(k)}x(k+1) = A_{\sigma(k)}x(k) + B_{\sigma(k)}u(k) \tag{1a}$$
$$y(k) = C_{\sigma(k)}x(k) \tag{1b}$$

where $k \in \mathbb{N}$ is the time instant; $x(k) \in \mathbb{R}^n$, $n \in \mathbb{N}$, is the state at time k; $y(k) \in \mathbb{R}^p, p \in \mathbb{N}$ is the output; $\sigma : \mathbb{N} \to \{1, 2, ..., \mathrm{p}\}$ is the switching signal determining which mode $\sigma(k)$ is active at time instant k; $E_i, A_i \in \mathbb{R}^{n\times n}$, $B_i \in \mathbb{R}^{n\times m}$ $C_i \in \mathbb{R}^{p\times n}$, $A_i \in \mathbb{R}^{p\times m}$, $i \in \{1, 2, \ldots, \mathrm{p}\}$; E_i may be singular.

For the considered switched systems, the switching signal is restricted to a fixed and known switching signal given via a fixed mode sequence $(\sigma_j)_{j\in\mathbb{N}}$ as follows (see also Fig. 1)

$$\sigma(k) = \sigma_j \text{ if } k \in [k_j^s, k_{j+1}^s),\ j \in \{0, 1, 2, ..., J\}, \tag{2}$$

where $k_j^s \in \mathbb{N}$ denote the switching times with $k_0^s = 0$, $k_{J+1}^s = K$, and $\sigma_j \in \{1, 2, ..., \mathrm{p}\}$. Here $K > 0$ denotes the final time of interest and $J > 0$ is the number of switches on the time interval $[0, K]$. Furthermore, note that the solution at $k = K$ is obtained from the last mode equation, and hence we only consider the equation (13) up to $k = K - 1$. In addition, the switching times k_j^s are assumed to be strictly increasing, i.e. $k_{j+1}^s > k_j^s$.

Define the reverse switching signal of σ as (see also Fig. 2)

$$\begin{aligned} \bar{\sigma}(k) = \sigma_j \text{ if } k \in [K - k_{j+1}^s, K - k_j^s), \\ j \in \{J, J-1, ..., 0\}, \end{aligned} \tag{3}$$

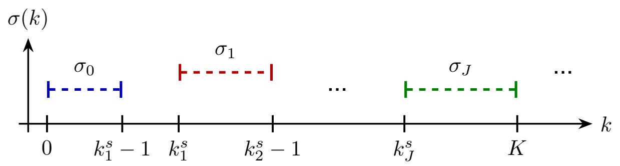

Fig. 1. The switching signal (2)

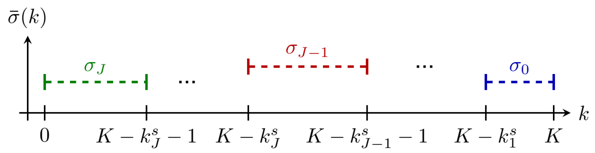

Fig. 2. The reversed switching signal (3)

The definition of duality we consider in this paper has the following properties (c.f. [22]):

- The dual of the dual is the original system
- The dual is an element of the same system class
- The classical duality between (some form of) reachability and (some form of) observability holds

III. PRELIMINARIES

For linear systems, it is well-known that

$$x(k+1) = Ax(k) + Bu(k) \tag{4a}$$
$$y(k) = Cx(k) \tag{4b}$$

is reachable (observable) on the time interval $[0, \infty)$ if and only if its dual system

$$x(k+1) = A^\top x(k) + C^\top u(k) \tag{5a}$$
$$y(k) = B^\top x(k) \tag{5b}$$

is observable (reachable) on $[0, \infty)$, see e.g. [20]. When a finite time interval of observation is considered, the reachability and observability notions should be precise (it does not matter on $[0, \infty)$). We recall the definitions as follows.

Definition 1 (Observability on $[0, K]$)**.** *A system is observable on $[0, K]$ if for all solutions (x^1, y^1) and (x^2, y^2) on $[0, K]$ the following implication holds:*

$$y^1_{[0,K]} = y^2_{[0,K]} \implies x^1_{[0,K]} = x^2_{[0,K]}. \tag{6}$$

Definition 2 (Reachability from Zero on $[0, K]$)**.** *A state $x_f \in \mathbb{R}^n$ is called* reachable from zero *on $[0, K]$ if with $x(0) = 0$ there exists an input sequence $(u(0), u(1), .., u(K))$ such that $x(K+1) = x_f$.* //

In the sense of the definition above, a reachable state can be reached within $(K+1)$ steps (in the classical sense within K steps). Now, we have that for ordinary linear systems, (A, B, C) is completely reachable (completely observable) on $[0, K]$ if and only if its dual $(A^\top, C^\top, B^\top)$ is completely observable (completely reachable). For short, we say reachable or observable. We highlight the main idea of the proof as follows. For the original system (A, B, C), a necessary and sufficient conditions of reachability and observability on $[0, K]$ are

$$\operatorname{rank}[B, AB, \ldots, A^K B] = n \text{ and } \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^K \end{bmatrix} = n$$

respectively. Meanwhile, for the dual system $(A^\top, C^\top, B^\top)$, a necessary and sufficient conditions of reachability and observability on $[0, K]$ are

$$\operatorname{rank}[C^\top, A^\top C^\top, \ldots, A^{\top K} C^\top] = n \iff \operatorname{rank}\begin{bmatrix} C \\ CA \\ \vdots \\ CA^K \end{bmatrix} = n$$

and

$$\operatorname{rank}\begin{bmatrix} B^\top \\ B^\top A^\top, \\ \vdots, \\ B^\top A^{\top K} \end{bmatrix} = n \iff \operatorname{rank}[B, AB, \ldots, A^K B] = n$$

respectively.

IV. (NONSWITCHED) SINGULAR LINEAR SYSTEMS

Consider first system (1) without switching and assume it is *strongly solvable (for short: solvable)*[1]:

$$(E, A, B, C) := \begin{cases} Ex(k+1) = Ax(k) + Bu(k) \\ \qquad y(k) = Cx(k) \end{cases} \tag{7}$$

i.e. it satisfies $\operatorname{im}[E^+A, E^+B] \subseteq \widehat{\mathcal{S}} \oplus \ker E$ where $\widehat{\mathcal{S}} := A^{-1}(\operatorname{im}[A, B])$ is the consistency space–the set of all initial values that have a unique solution on any time interval $[0, k_1]$, $\forall k_1 \in \mathbb{N}$ [24]. System (7) together with its consistency space is denoted by the tuple $(E, A, B, C, \widehat{\mathcal{S}})$.

One may consider the "naive" dual $(E^\top, A^\top, C^\top, B^\top)$. However, note that the solvability of an InhSLS does not imply the solvability of its naive dual; this is shown by Example 1.

Example 1. *Consider system* (7) *with*

$$(E, A, B, C) = \left(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]^\top\right),$$

[1] System (7) is strongly solvable if, for all $k_1 \in \mathbb{N}$, for all $x_0 \in \widehat{\mathcal{S}}$ and all input sequences $(u(0), u(1), ..., u(k_1 - 1))$, there exists a unique solution of (7) considered on $[0, k_1]$ with $x(0) = x_0$ [23, D. 2.1].

which is solvable with its consistency space $\widehat{\mathcal{S}} = \operatorname{im}\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$. Its naive dual, $(E^\top, A^\top, C^\top, B^\top)$ is not solvable as its corresponding consistency space $\widehat{\mathcal{S}}^\top = \mathbb{R}^2$, and thus $\widehat{\mathcal{S}}^\top \cap \ker E^\top \neq \{0\}$ and the solvability condition is not satisfied.

From [24, Lemma 3.2], every solution of (7) with $x(0) = x_0 \in \widehat{\mathcal{S}}$ satisfies the following surrogate ordinary system with the same initial condition:

$$(I, \Theta, \Gamma, C, \widehat{\mathcal{S}}) := \begin{cases} x(k+1) = \Theta x(k) + \Gamma u(k) \\ \quad\; y(k) = Cx(k) \end{cases} \tag{8}$$

which is an ordinary system with the initial condition is restricted to a certain subspace i.e. $x(0) = x_0 \in \widehat{\mathcal{S}}$. This also means that for every solvable InhSLS, there is an equivalent ordinary linear system considered with initial values taken only from the consistency space of the InhSLS. Now, define the dual of the surrogate InhSLS (8) as follows:

$$(I, \Theta^\top, C^\top, \Gamma^\top, \widehat{\mathcal{S}}) := \begin{cases} x(k+1) = \Theta^\top x(k) + C^\top u(k) \\ \quad\; y(k) = \Gamma^\top x(k) \end{cases} \tag{9}$$

which is also an ordinary linear system considered with initial values in $\widehat{\mathcal{S}}$. After the dual of the surrogate system is derived, the next question is whether this dual has its corresponding singular system; this remains an open problem.

The following lemma is the main result of this part that says that system (9) is indeed the dual of system (8).

Lemma 2. *The InhSLS $(E, A, B, C, \widehat{\mathcal{S}})$ or its surrogate system $(I, \Theta, \Gamma, C, \widehat{\mathcal{S}})$ is observable (reachable) on $[0, K]$, $K \in \mathbb{N}$ if and only if its dual $(I, \Theta^\top, C^\top, \Gamma^\top, \widehat{\mathcal{S}})$ is reachable (observable) on $[0, K]$.* //

Proof. For the original system (7) or the surrogate system (8), a necessary and sufficient condition of reachability on $[0, K]$ is

$$\widehat{\mathcal{S}} \cap \operatorname{im}[\Gamma, \Theta\Gamma, \dots, \Theta^K\Gamma] = \widehat{\mathcal{S}}. \tag{10}$$

and a necessary and sufficient condition of observability on $[0, K]$ is

$$\widehat{\mathcal{S}} \cap \ker \begin{bmatrix} C \\ C\Theta \\ \vdots \\ C\Theta^K \end{bmatrix} = \{0\}. \tag{11}$$

Now, for the dual surrogate system (9), a necessary and sufficient condition of complete reachability on $[0, K]$ is

$$\widehat{\mathcal{S}} \cap \operatorname{im}[C^\top, \Theta^\top C^\top, \dots, \Theta^{\top K} C^\top] = \widehat{\mathcal{S}}$$

$$\iff \widehat{\mathcal{S}} \cap \left(\ker \begin{bmatrix} C \\ C\Theta \\ \vdots \\ C\Theta^K \end{bmatrix}\right)^\perp = \widehat{\mathcal{S}}$$

$$\iff \widehat{\mathcal{S}} \subseteq \left(\ker \begin{bmatrix} C \\ C\Theta \\ \vdots \\ C\Theta^K \end{bmatrix}\right)^\perp$$

$$\iff \widehat{\mathcal{S}} \cap \ker \begin{bmatrix} C \\ C\Theta \\ \vdots \\ C\Theta^K \end{bmatrix} = \{0\} \tag{12}$$

which is equivalent to the observability condition (11). Finally, a necessary and sufficient condition of observability on $[0, K]$ is

$$\widehat{\mathcal{S}} \cap \ker \begin{bmatrix} \Gamma^\top \\ \Gamma^\top \Theta^\top \\ \vdots \\ \Gamma^\top \Theta^{\top K} \end{bmatrix} = \{0\}$$

$$\iff \widehat{\mathcal{S}} \cap \operatorname{im}[\Gamma, \Theta\Gamma, \dots, \Theta^K\Gamma]^\perp = \{0\}$$

$$\iff \widehat{\mathcal{S}} \subseteq \operatorname{im}[\Gamma, \Theta\Gamma, \dots, \Theta^K\Gamma] = \{0\}$$

$$\iff \widehat{\mathcal{S}} \cap \operatorname{im}[\Gamma, \Theta\Gamma, \dots, \Theta^K\Gamma] = \widehat{\mathcal{S}}$$

which is equivalent to the reachability condition (10). This completes the proof. □

V. (Ordinary) Linear Switched Systems

Consider now the nonsingular version of (1), i.e., the Inhomogeneous Linear Switched System (InhLSS)

$$x(k+1) = A_{\sigma(k)} x(k) + B_{\sigma(k)} u(k) \tag{13a}$$

$$y(k) = C_{\sigma(k)} x(k) \tag{13b}$$

and define its dual switched system

$$x(k+1) = A^\top_{\bar{\sigma}(k)} x(k) + C^\top_{\bar{\sigma}(k)} u(k) \tag{14a}$$

$$y(k) = B^\top_{\bar{\sigma}(k)} x(k) \tag{14b}$$

where $\bar{\sigma}$ is defined as (3).

Lemma 1. *The InhLSS (13) is observable (reachable) w.r.t. the switching signal σ on $[0, K]$ if and only if its dual (14) is reachable (observable) w.r.t. the (reverse) switching signal $\bar{\sigma}$ on $[0, K]$.*

Proof. For the original system (13) under a fixed switching signal σ on $[0, K]$, a necessary and sufficient condition of reachability is

$$\begin{aligned} \operatorname{rank}[&A_{\sigma_J}^{K-k_J^s} A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s} \cdots A_2^{k_3^s - k_2^s} A_1^{k_2^s - k_1^s} \\ &[B_{\sigma_0}, A_{\sigma_0} B_{\sigma_0}, \dots, A_{\sigma_0}^{k_1^s - 1} B_{\sigma_0}], \\ &A_{\sigma_J}^{K-k_J^s} A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s} \cdots A_3^{k_4^s - k_3^s} A_2^{k_3^s - k_2^s} \\ &[B_{\sigma_1}, A_{\sigma_1} B_{\sigma_1}, \dots, A_{\sigma_1}^{k_2^s - k_1^s - 1} B_{\sigma_1}], \dots, \\ &[B_{\sigma_J}, A_{\sigma_J} B_{\sigma_J}, \dots, A_{\sigma_J}^{K-k_J^s} B_{\sigma_J}]] = n. \end{aligned} \tag{15}$$

and a necessary and sufficient condition of observability is

$$\operatorname{rank} \begin{bmatrix} C_{\sigma_0} \\ C_{\sigma_0} A_{\sigma_0} \\ \vdots \\ C_{\sigma_0} A_{\sigma_0}^{k_1^s - 1} \\ C_{\sigma_1} A_{\sigma_0}^{k_1^s} \\ C_{\sigma_1} A_{\sigma_1} A_{\sigma_0}^{k_1^s} \\ \vdots \\ C_{\sigma_1} A_{\sigma_1}^{k_2^s - k_1^s - 1} A_{\sigma_0}^{k_1^s} \\ \vdots \\ C_{\sigma_J} A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s} \cdots A_{\sigma_1}^{k_2^s - k_1^s} A_{\sigma_0}^{k_1^s} \\ C_{\sigma_J} A_{\sigma_J} A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s} \cdots A_{\sigma_1}^{k_2^s - k_1^s} A_{\sigma_0}^{k_1^s} \\ \vdots \\ C_{\sigma_J} A_{\sigma_J}^{K-k_J^s} A_{\sigma_{J-1}}^{k_J^s - k_{J-1}^s} \cdots A_{\sigma_1}^{k_2^s - k_1^s} A_{\sigma_0}^{k_1^s} \end{bmatrix} = n. \tag{16}$$

Now, for the dual system (14) under the (reverse) fixed switching signal $\bar{\sigma}$ on $[0, K]$, a necessary and sufficient condition of complete reachability is

$$\begin{gathered}\operatorname{rank}[A_0^{\top k_1^s} A_1^{\top k_2^s - k_1^s} \cdots A_{\sigma_{J-1}}^{\top k_J^s - k_{J-1}^s} \\ [C_{\sigma_J}^\top, A_{\sigma_J}^\top C_{\sigma_J}^\top, \ldots, A_{\sigma_J}^{\top K - k_J^s} C_{\sigma_J}^\top], \ldots, \\ A_{\sigma_0}^{\top k_1^s} [C_{\sigma_1}^\top, A_{\sigma_1}^\top C_{\sigma_1}^\top, \ldots, A_{\sigma_1}^{\top k_2^s - k_1^s} C_{\sigma_1}^\top] \\ [C_{\sigma_0}^\top, A_{\sigma_0}^\top C_{\sigma_0}^\top, \ldots, A_{\sigma_0}^{\top k_1^s - 1} C_{\sigma_0}^\top]] = n\end{gathered} \tag{17}$$

which is equivalent to condition (16). Finally, a necessary and sufficient condition of observability is

$$\operatorname{rank}\begin{bmatrix} B_{\sigma_J}^\top \\ B_{\sigma_J}^\top A_{\sigma_J}^\top \\ \vdots \\ B_{\sigma_J}^\top A_{\sigma_J}^{\top K - k_J^s - 1} \\ B_{\sigma_{J-1}}^\top A_{\sigma_J}^{\top K - k_J^s} \\ B_{\sigma_{J-1}}^\top A_{\sigma_{J-1}}^\top A_{\sigma_J}^{\top K - k_J^s} \\ \vdots \\ B_{\sigma_{J-1}}^\top A_{\sigma_{J-1}}^{\top k_J^s - k_{J-1}^s} A_{\sigma_J}^{\top K - k_J^s} \\ \vdots \\ B_{\sigma_0}^\top A_{\sigma_1}^{\top k_2^s - k_1^s} A_{\sigma_J}^{K - k_J^s} \\ B_{\sigma_0}^\top A_{\sigma_0} A_{\sigma_1}^\top k_2^s - k_1^s \cdots A_{\sigma_J}^{\top K - k_J^s} \\ \vdots \\ B_{\sigma_0}^\top A_{\sigma_0}^{\top k_1^s} A_{\sigma_1}^{\top k_2^s - k_1^s} \ldots A_{\sigma_J}^{\top K - k_J^s} \end{bmatrix} = n \tag{18}$$

which is equivalent to condition (15). $\square$

VI. Singular Linear Switched Systems

Now, consider the InhSLSS (1) and assume that it is solvable for the given switching signal. From [24, Theorem 3.6], every solution of (1) with $x(0) = x_0 \in \widehat{\mathcal{S}}_{\sigma_0}$ satisfies the following surrogate ordinary switched system

$$\begin{gathered}(I, \Theta_\sigma, \Gamma_\sigma, C_\sigma, \widehat{\mathcal{S}}_{\sigma_0}) := \\ \begin{cases} x(k+1) = \Theta_{\sigma(k+1),\sigma(k)} x(k) + \Gamma_{\sigma(k+1),\sigma(k)} u(k) \\ \quad y(k) = C_{\sigma(k)} x(k) \end{cases}\end{gathered} \tag{19}$$

where $\Theta_{i,j} := \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+ A_j$ and $\Gamma_{i,j} := \Pi_{\widehat{\mathcal{S}}_i}^{\ker E_j} E_j^+ B_j$. Note that reversing the switching signal, as in the dual systems of (ordinary) linear switched systems discussed in Section II, does not work in general since $\Theta_{i,j} \neq \Theta_{j,i},\ i \neq j$ as well as $\Gamma_{i,j} \neq \Gamma_{j,i}$. We now propose to see the surrogate switched system above with the given switching signal as a surrogate time-varying system:

$$\begin{gathered}(I, \Theta(k), \Gamma(k), C(k), \widehat{\mathcal{S}}_{\sigma_0}) := \\ \begin{cases} x(k+1) = \Theta(k) x(k) + \Gamma(k) u(k) \\ \quad y(k) = C(k) x(k) \end{cases}\end{gathered} \tag{20}$$

with $\Theta(k) = \Theta_{\sigma(k+1),\sigma(k)}$, $\Gamma(k) = \Gamma_{\sigma(k+1),\sigma(k)}$, and $C(k) = C_{\sigma(k)}$. Now, define the dual of the surrogate time-varying system (20) as follows:

$$\begin{gathered}(I, \Theta(K-k)^\top, C(K-k)^\top, \Gamma(K-k)^\top, \widehat{\mathcal{S}}_{\sigma_J}) \\ := \begin{cases} x(k+1) = \Theta(K-k)^\top x(k) + C(K-k)^\top u(k) \\ \quad y(k) = \Gamma(K-k)^\top x(k). \end{cases}\end{gathered} \tag{21}$$

The dual system above is derived by reversing the "switching signal" i.e. the order of the system's matrices w.r.t. the time. Again, finding the corresponding InhSLSS for this dual still remains open.

The following lemma is the main result of this section, which says that the dual above is indeed the dual of the surrogate system (20).

Lemma 1. *The InhSLSS* $(E_{\sigma(k)}, A_{\sigma(k)}, B_{\sigma(k)}, C_{\sigma(k)}, \widehat{\mathcal{S}}_{\sigma_0})$ *via its surrogate time-varying system* $(I, \Theta(k), \Gamma(k), C(k), \widehat{\mathcal{S}}_{\sigma_0})$ *is observable (reachable) on* $[0, K]$, $K \in \mathbb{N}$ *if and only if its dual time-varying system* $(I, \Theta(K-k)^\top, C(K-k)^\top, \Gamma(K-k)^\top, \widehat{\mathcal{S}}_{\sigma_J})$ *is reachable (observable) on* $[0, K]$.

Proof. For the original InhSLSS (1) or its surrogate time-varying system (20), a necessary and sufficient condition of complete reachability on $[0, K]$ is

$$\begin{gathered}\widehat{\mathcal{S}}_{\sigma_J} \cap \operatorname{im}[\Theta(K)\Theta(K-1)\cdots\Theta(1)\Gamma(0), \\ \Theta(K)\Theta(K-1)\cdots\Theta(2)\Gamma(1), \ldots, \Theta(K)\Gamma(K-1), \\ \Gamma(K)] = \widehat{\mathcal{S}}_{\sigma_J}.\end{gathered} \tag{22}$$

and a necessary and sufficient condition of observability on $[0, K]$ is

$$\widehat{\mathcal{S}}_{\sigma_0} \cap \ker \begin{bmatrix} C(0) \\ C(1)\Theta(0) \\ C(2)\Theta(1)\Theta(0) \\ \vdots \\ C(K)\Theta(K-1)\cdots\Theta(1)\Theta(0) \end{bmatrix} = \{0\}. \tag{23}$$

For the dual surrogate time-varying system (21), a necessary and sufficient condition of complete reachability on $[0, K]$ is

$$\begin{gathered}\widehat{\mathcal{S}}_{\sigma_0} \cap \operatorname{im}[C(0)^\top, \Theta(0)^\top C(1)^\top, \Theta(0)^\top \Theta(1)^\top C(2)^\top, \\ \ldots, \Theta(0)^\top \Theta(1)^\top \cdots \Theta(K-1)^\top C(K)^\top] = \widehat{\mathcal{S}}_{\sigma_0}\end{gathered}$$

$$\iff \widehat{\mathcal{S}}_{\sigma_0} \cap \left(\ker \begin{bmatrix} C(0) \\ C(1)\Theta(0) \\ C(2)\Theta(1)\Theta(0) \\ \vdots \\ C(K)\Theta(K-1)\cdots\Theta(1)\Theta(0) \end{bmatrix} \right)^\perp = \widehat{\mathcal{S}}_{\sigma_0}$$

$$\iff \widehat{\mathcal{S}}_{\sigma_0} \subseteq \left(\ker \begin{bmatrix} C(0) \\ C(1)\Theta(0) \\ C(2)\Theta(1)\Theta(0) \\ \vdots \\ C(K)\Theta(K-1)\cdots\Theta(1)\Theta(0) \end{bmatrix} \right)^\perp$$

$$\iff \widehat{\mathcal{S}}_{\sigma_0} \cap \ker \begin{bmatrix} C(0) \\ C(1)\Theta(0) \\ C(2)\Theta(1)\Theta(0) \\ \vdots \\ C(K)\Theta(K-1)\cdots\Theta(1)\Theta(0) \end{bmatrix} = \{0\}$$

which is equivalent to condition (23), and a necessary and sufficient condition of observability on $[0, K]$ is

$$\begin{aligned}
&\widehat{\mathcal{S}}_{\sigma_J} \cap \ker \begin{bmatrix} \Gamma(K)^\top \\ \Gamma(K-1)^\top \Theta(K)^\top \\ \vdots \\ \Gamma(1)^\top \Theta(2)^\top \Theta(3)^\top \cdots \Theta(K-1)\Theta(K)^\top \\ \Gamma(0)^\top \Theta(1)^\top \Theta(2)^\top \cdots \Theta(K-1)\Theta(K)^\top \end{bmatrix} = \{0\} \\
&\iff \widehat{\mathcal{S}}_{\sigma_J} \cap \operatorname{im}[\Gamma(K), \Theta(K)\Gamma(K-1), \dots, \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(3)\Theta(2)\Gamma(1), \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(2)\Theta(1)\Gamma(0)]^\perp = \{0\} \\
&\iff \widehat{\mathcal{S}}_{\sigma_J} \subseteq \operatorname{im}[\Gamma(K), \Theta(K)\Gamma(K-1), \dots, \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(3)\Theta(2)\Gamma(1), \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(2)\Theta(1)\Gamma(0)] = \{0\} \\
&\iff \widehat{\mathcal{S}}_{\sigma_J} \cap \operatorname{im}[\Gamma(K), \Theta(K)\Gamma(K-1), \dots, \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(3)\Theta(2)\Gamma(1), \\
&\quad \Theta(K)\Theta(K-1)\cdots\Theta(2)\Theta(1)\Gamma(0)] = \widehat{\mathcal{S}}_{\sigma_J}
\end{aligned}$$

which is equivalent to condition (22). This completes the proof. □

VII. Illustrative Example

One example of system (1) is the switched Leontief economic model (see e.g. [25] for the original form of the equation). Consider that there are two modes $\{1, 2\}$ with matrices (see [23] for the detailed derivation) $B_i = C_i = I$ for $i = 1, 2$,

$$E_1 = \begin{bmatrix} 0.30 & 0.40 & 0.45 \\ 0 & 0 & 0 \\ 0.60 & 0.80 & 0.90 \end{bmatrix}, \qquad E_2 = E_1$$
$$A_1 = \begin{bmatrix} 1.00 & 0.10 & 0.15 \\ -0.40 & 0.90 & -0.50 \\ 0.30 & 0.30 & 1.70 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1.00 & 0.10 & 0.15 \\ -0.40 & 0.90 & -0.50 \\ 0.60 & 0.80 & 1.90 \end{bmatrix},$$

and

$$\ker E_1 = \operatorname{im} \begin{bmatrix} 1.00 & 1.13 \\ -0.75 & 0.00 \\ 0.00 & -0.75 \end{bmatrix}, \quad \ker E_2 = \operatorname{im} \begin{bmatrix} 1.00 & 1.13 \\ -0.75 & 0.00 \\ 0.00 & -0.75 \end{bmatrix},$$
$$\mathcal{S}_1 = \operatorname{im} \begin{bmatrix} 1.00 \\ 1.08 \\ 1.14 \end{bmatrix}, \qquad \mathcal{S}_2 = \operatorname{im} \begin{bmatrix} 1.00 \\ 0.77 \\ 0.59 \end{bmatrix},$$

The corresponding surrogate system's matrices both for each mode and for the the mode transitions $1 \to 2$ and $2 \to 1$ with

$$E_1^+ = \begin{bmatrix} -0.12 & 0.00 & 0.21 \\ 0.13 & 0.00 & 0.00 \\ 0.10 & 0.00 & -0.05 \end{bmatrix}$$

are given by

$$\Theta_{1,1} = \begin{bmatrix} 0.12 & 0.12 & 0.68 \\ 0.13 & 0.13 & 0.74 \\ 0.14 & 0.14 & 0.78 \end{bmatrix} \qquad \Theta_{2,2} = \begin{bmatrix} 0.34 & 0.46 & 1.09 \\ 0.26 & 0.35 & 0.84 \\ 0.20 & 0.27 & 0.64 \end{bmatrix},$$
$$\Theta_{2,1} = \begin{bmatrix} 0.17 & 0.17 & 0.97 \\ 0.13 & 0.13 & 0.75 \\ 0.10 & 0.10 & 0.57 \end{bmatrix}, \quad \Theta_{1,2} = \begin{bmatrix} 0.24 & 0.32 & 0.76 \\ 0.26 & 0.35 & 0.82 \\ 0.27 & 0.37 & 0.87 \end{bmatrix},$$
$$\Gamma_{1,1} = \begin{bmatrix} 0.00 & 0.00 & 0.40 \\ 0.00 & 0.00 & 0.43 \\ 0.00 & 0.00 & 0.46 \end{bmatrix}, \quad \Gamma_{2,2} = \begin{bmatrix} 0.00 & 0.00 & 0.57 \\ 0.00 & 0.00 & 0.44 \\ 0.00 & 0.00 & 0.34 \end{bmatrix},$$
$$\Gamma_{2,1} = \begin{bmatrix} 0.00 & 0.00 & 0.57 \\ 0.00 & 0.00 & 0.44 \\ 0.00 & 0.00 & 0.34 \end{bmatrix}, \quad \Gamma_{1,2} = \begin{bmatrix} 0.00 & 0.00 & 0.40 \\ 0.00 & 0.00 & 0.43 \\ 0.00 & 0.00 & 0.46 \end{bmatrix}$$
.

The dual systems for the two individual modes are then given by

$$\begin{aligned} x(k+1) &= \Theta_{1,1}^\top x(k) + u(k) \\ y(k) &= \Gamma_{1,1}^\top x(k) \end{aligned}$$

and

$$\begin{aligned} x(k+1) &= \Theta_{2,2}^\top x(k) + u(k) \\ y(k) &= \Gamma_{2,2}^\top x(k) \end{aligned}$$

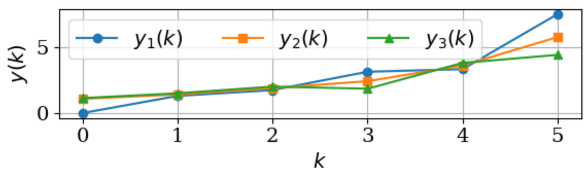

Fig. 3. Outputs of the original system

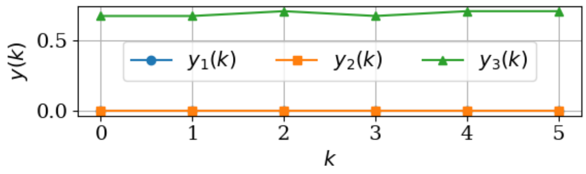

Fig. 4. Outputs of the dual system ($y_1(k) = y_2(k) \equiv 0$)

respectively. Now, consider the switching signal $\sigma(k) = \{(0,1), (1,1), (2,2), (3,1), (4,2), (5,2)\}$. The original switched surrogate system and its corresponding dual system are

$$x(k+1) = \begin{cases} \Theta_{1,1} x(k) + \Gamma_{1,1} u(k), & k = 0, 1; \\ \Theta_{2,1} x(k) + \Gamma_{2,1} u(k), & k = 2; \\ \Theta_{1,2} x(k) + \Gamma_{1,2} u(k), & k = 3; \\ \Theta_{2,2} x(k) + \Gamma_{2,2} u(k), & k = 4, 5; \end{cases}$$
$$y(k) = x(k), \quad k = 0, 1, \dots, 5.$$

and

$$x(k+1) = \begin{cases} \Theta_{1,1}^\top x(k) + u(k), & k = 0, 1; \\ \Theta_{2,1}^\top x(k) + u(k), & k = 2; \\ \Theta_{1,2}^\top x(k) + u(k), & k = 3; \\ \Theta_{2,2}^\top x(k) + u(k), & k = 4, 5; \end{cases}$$
$$y(k) = \begin{cases} \Gamma_{2,2}^\top u(k), & k = 0, 1; \\ \Gamma_{1,2}^\top u(k), & k = 2; \\ \Gamma_{2,1}^\top u(k), & k = 3; \\ \Gamma_{1,1}^\top u(k), & k = 4, 5; \end{cases}$$

respectively. With initial value $x(0) = (1.00, 1.08, 1.14)^\top$ for the original system and $x(0) = (1.00, 0.77, 0.59)^\top$ for the dual system, and constant input $u(k) = (0, 0, 1)^\top$, the outputs are illustrated in Fig. 3 and 4.

VIII. Summary and Future Works

This paper investigated the duality between reachability and observability for discrete-time singular linear switched systems. By introducing suitable surrogate representations, the structural relationship between these two fundamental system properties was analyzed. In particular, it was shown that for the surrogate time-varying representation of the considered systems, reachability and observability exhibit a clear dual relationship under time reversal and matrix transposition. These results extend classical duality principles known for ordinary linear systems to a broader class of hybrid systems that involve both switching dynamics and singular (descriptor) structures.

Several directions for future research remain open. In particular, finding the corresponding singular switched system representation for the dual surrogate systems remains an open problem. Establishing this would provide a fully duality framework. Furthermore, extending the presented results to more general switching signals, uncertain systems, or systems with constraints represents an interesting direction for further investigation. A comprehensive comparison between the discrete- and continuous-time domains also remains open for investigation. Finally, the obtained duality framework may support the development of new methods for observer design, controller synthesis, and system identification in complex hybrid systems.

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